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Technical Report No. 32-79

The Motion of a Satellite Under the Influence of a Constant Normal Thrust

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ABSTRACT

The motion of a satellite in a central inverse-square-law force field, subject to a constant thrust normal to the instantaneous plane of motion, is considered. An exact closed-form solution is given for the case of initially circular orbits, and approximate closed-form solutions are found for initially elliptical orbits, using the methods of Kryloff and Bogoliuboff, and Poincaré.

I. INTRODUCTION

The motion of a satellite which is in a central inverse-square-law force field and is subject to a constant thrust normal to the instantaneous plane of its motion is considered. The assumption is made that the mass change of the satellite is negligible. In Sec. II the equations of motion and the mathematical framework are developed. The analysis in Sec. III, in which an exact closed-form solution is given for the case of a satellite originally in circular motion, yields rather surprising results. In Sec. IV, the Kryloff–Bogoliuboff averaging process (Ref. 1) is applied to the case of elliptical orbits in order to obtain another closed-form solution; in Sec. V an approximate closed-form solution is given for the case of elliptical orbits of small eccentricity.

II. EQUATIONS OF MOTION

In Fig. 1 the rectangular coordinate system $OXYZ$ is an inertial frame of reference, where O is the origin of an inverse-square-law force field acting on the satellite. The $Oxyz$ rectangular coordinate system is chosen so that the satellite is always on the positive x-axis at a distance r from the origin, with the z-axis in the direction of the angular momentum of the satellite. Under these circumstances the instantaneous velocity vector of the satellite lies in the x–y plane. The angular velocity of the $Oxyz$ frame of reference relative to the inertial frame $OXYZ$ will be designated by $\boldsymbol{\omega} = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$, with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ unit vectors along the positive x, y, and z axes, respectively. The plane defined by the instantaneous position and velocity vectors of the satellite is referred to as the instantaneous plane of the motion. The thrust per unit mass on the satellite will be given by $\mathbf{W} = W\mathbf{k}$, where W is a constant.

Using the dot notation to denote time derivatives, the equation of motion is

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} + W\mathbf{k} \tag{1}$$

From $\mathbf{r} = r\mathbf{i}$ and $d\mathbf{i}/dt = \boldsymbol{\omega} \times \mathbf{i}$, it follows that

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{i} + r\omega_z \mathbf{j} - r\omega_y \mathbf{k} \tag{2}$$

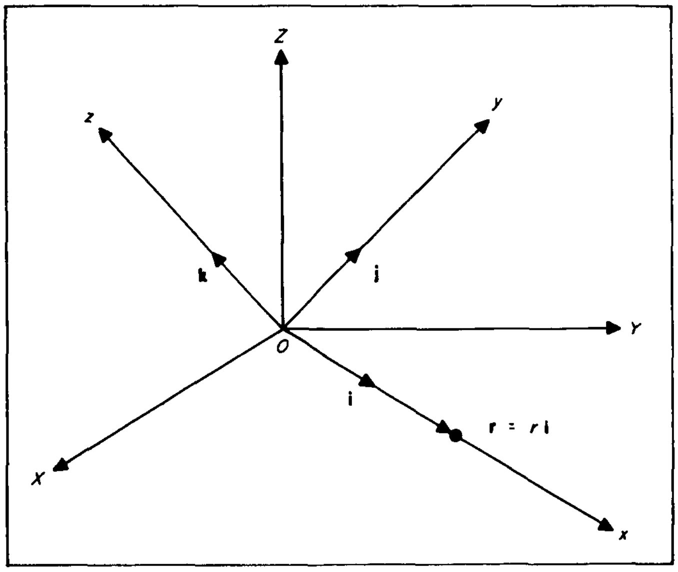

Fig. 1. Frames of reference associated with the satellite

Since $\mathbf{v}$ lies in the x–y plane, of necessity

$$\omega_y \equiv 0 \tag{3}$$

and Eq. (2) becomes

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{i} + r\omega_z \mathbf{j} \tag{4}$$

A further differentiation yields

$$\dot{\mathbf{v}} = \ddot{\mathbf{r}} = (\ddot{r} - r\omega_z^2)\,\mathbf{i} + \frac{1}{r}\frac{d}{dt}(r^2\omega_z)\,\mathbf{j} + r\omega_x\omega_z\mathbf{k} \tag{5}$$

From Eq. (1) and (5) we obtain

$$\begin{aligned} \ddot{r} - r\omega_z^2 &= -\frac{\mu}{r^2} \\ \frac{d}{dt}(r^2\omega_z) &= 0 \\ r\omega_x\omega_z &= W \end{aligned} \tag{6}$$

From Eq. (6) it follows that the magnitude of the z-component of the angular momentum, given by $h = r^2\omega_z$, remains constant. From $d\mathbf{i}/dt = \boldsymbol{\omega} \times \mathbf{i}$, etc., we can write Eq. (6) in the form

$$\begin{aligned} \ddot{r} - \frac{h^2}{r^3} &= -\frac{\mu}{r^2} \\ \frac{d\mathbf{i}}{dt} &= \frac{h}{r^2}\mathbf{j} \\ \frac{d\mathbf{j}}{dt} &= -\frac{h}{r^2}\mathbf{i} + \frac{rW}{h}\mathbf{k} \\ \frac{d\mathbf{k}}{dt} &= -\frac{rW}{h}\mathbf{j} \end{aligned} \tag{7}$$

The unknowns are $\mathbf{i}, \mathbf{j}, \mathbf{k}$, and r. Once r and $\mathbf{i}$ are determined, the motion is completely specified. It is interesting to note that the vector equations of Eq. (7) are functionally the Frenet–Serret formulas of differential geometry (Ref. 2).

If we define θ by the equation $r^2\dot{\theta} = h$ and let $u = 1/r$, Eq. (7) can be simplified to

$$
\begin{aligned}
\frac{d^2u}{d\theta^2} + u &= \frac{\mu}{h^2} \\
\frac{d\mathbf{i}}{d\theta} &= \mathbf{j} \\
\frac{d\mathbf{j}}{d\theta} &= -\mathbf{i} + \frac{W}{h^2u^3}\mathbf{k} \\
\frac{d\mathbf{k}}{d\theta} &= -\frac{W}{h^2u^3}\mathbf{j}
\end{aligned} \tag{8}
$$

The solution to the first equation of the set of Eq. (8) is

$$u = \frac{1}{r} = \frac{\mu}{h^2}\left[1 + e\cos(\theta - \phi)\right] \tag{9}$$

with e and ϕ constants of integration determined from initial conditions. Since θ is determined only up to an additive constant, there is no loss in generality in assuming $\phi \equiv 0$, which we henceforth do. In the r–θ plane, Eq. (9) represents a conic section of eccentricity e which is independent of the normal thrust W.

For the special case in which $W \equiv 0$, we obtain

$$
\begin{aligned}
\mathbf{i} &= \mathbf{i}_0 \cos\theta + \mathbf{j}_0 \sin\theta \\
\mathbf{j} &= -\mathbf{j}_0 \sin\theta + \mathbf{j}_0 \cos\theta \\
\mathbf{k} &= \mathbf{k}_0
\end{aligned} \tag{10}
$$

where $\mathbf{i}_0$, $\mathbf{j}_0$, and $\mathbf{k}_0$ are initial vectors at $\theta = 0$. It is apparent that the orbital plane is fixed in inertial space ($\mathbf{k} = \mathbf{k}_0$) and that the satellite traverses a conic section (one-body problem).

III. CIRCULAR ORBITS

For the following specific set of initial conditions (at $t = 0$):

$$
\begin{aligned}
r &= a \\
\dot{r} &= 0 \\
\dot{\theta} &= \left(\frac{\mu}{a^3}\right)^{1/2}
\end{aligned}
$$

the motion in the r–θ plane will be circular, with $r = a = h^2/\mu =$ constant. The solutions to Eq. (7) in this case are

$$
\begin{aligned}
\mathbf{i} &= \mathbf{A} + \mathbf{B}\cos\nu t + \mathbf{C}\sin\nu t \\
\mathbf{j} &= \frac{\nu}{\alpha}\left[-\mathbf{B}\sin\nu t + \mathbf{C}\cos\nu t\right] \\
\mathbf{k} &= \frac{1}{\alpha\beta}\left[\alpha^2\mathbf{A} - \beta^2\mathbf{B}\cos\nu t - \beta^2\mathbf{C}\sin\nu t\right]
\end{aligned} \tag{11}
$$

where

$$
\begin{aligned}
\alpha &= \frac{h}{a^2} & \mathbf{A} &= \frac{1}{\nu^2}\left[\beta^2\mathbf{i}_0 + \alpha\beta\mathbf{k}_0\right] \\
\beta &= \frac{aW}{h} & \mathbf{B} &= \frac{1}{\nu^2}\left[\alpha^2\mathbf{i}_0 - \alpha\beta\mathbf{k}_0\right] \\
\nu &= (\alpha^2 + \beta^2)^{1/2} & \mathbf{C} &= \frac{\alpha}{\nu}\mathbf{j} \\
h^2 &= \mu a
\end{aligned} \tag{12}
$$

and $\mathbf{i}_0$, $\mathbf{j}_0$, and $\mathbf{k}_0$ are determined from the initial position and velocity of the satellite. (Not to be confused with $\mathbf{i}_0$, $\mathbf{j}_0$, and $\mathbf{k}_0$ of Sec. II.)

From Eq. (12) it follows that **A**, **B**, and **C** are mutually perpendicular. The position of the satellite at any time t is given by $\mathbf{r} = a\mathbf{i}$, so that the satellite moves in a fixed plane normal to **A**.

From

$$\mathbf{B}\cdot\mathbf{B} = \mathbf{C}\cdot\mathbf{C} = \frac{\alpha^2}{\nu^2} = \frac{\dfrac{\mu^2}{a^4}}{W^2 + \dfrac{\mu^2}{a^4}} \tag{13}$$

it follows that the motion is circular with center at $a\mathbf{A}$, of radius

$$a_1 = a\frac{\alpha}{\nu} = a\frac{\dfrac{\mu}{a^2}}{\left(W^2 + \dfrac{\mu^2}{a^4}\right)^{1/2}} < a \tag{14}$$

and frequency

$$\nu = \frac{a}{a_1}\left(\frac{\mu}{a^3}\right)^{1/2} > \left(\frac{\mu}{a^3}\right)^{1/2} \tag{15}$$

If the motion is initially in the X–Y plane, then the plane of the motion is at a distance

$$d = a|\mathbf{A}| = \frac{a\beta}{\nu} = \frac{aW}{\left(W^2 + \frac{\mu^2}{a^4}\right)^{1/2}} < a \tag{16}$$

from the origin, and it is inclined at an angle

$$\cos^{-1} \frac{W \frac{\mu}{a^2}}{W^2 + \frac{\mu^2}{a^4}}$$

to the X–Y plane.

At any time t, the height of the satellite above the X–Y plane is given by

$$H = \mathbf{r} \cdot \mathbf{k}_0 = a\mathbf{i} \cdot \mathbf{k}_0 = a \frac{\alpha\beta}{\nu^2} (1 - \cos \nu t) \tag{17}$$

and the maximum height occurs at $\nu t = \pi$ with

$$H_{\max} = 2a \frac{\alpha\beta}{\nu^2} = 2a \frac{W \frac{\mu}{a^2}}{W^2 + \frac{\mu^2}{a^4}} \tag{18}$$

The absolute maximum height from the X–Y plane is achieved when $W = \mu/a^2$, in which case the height is $H = a$, the plane of the motion is inclined to the X–Y plane at an angle of 45 deg, and the time to achieve this height is

$$T = \frac{\pi}{\nu} = \pi \left(\frac{a}{2W}\right)^{1/2} = \pi \left(\frac{a}{2}\right)^{1/2} \left(\frac{\mu}{a^2}\right)^{-1/2}$$

IV. ELLIPTICAL ORBITS

If the initial motion is elliptical, $0 < e < 1$, and if the quantity

$$\kappa = \frac{Wb^4}{\mu^3} = \frac{W}{\frac{\mu}{a^2}} (1 - e^2)^2$$

is sufficiently small, the method of Kryloff–Bogoliuboff (Ref. 1) is applicable. Upon differentiating, Eq. (8) becomes

$$\begin{aligned} \frac{d^2\mathbf{i}}{d\theta^2} + \mathbf{i} &= \frac{\kappa}{(1 + e\cos\theta)^3} \mathbf{k} \\ \frac{d\mathbf{k}}{d\theta} &= -\frac{\kappa}{(1 + e\cos\theta)^3} \frac{d\mathbf{i}}{d\theta} \end{aligned} \tag{19}$$

We assume a solution in the form

$$\begin{aligned} \mathbf{i} &= \mathbf{A}\cos\theta + \mathbf{B}\sin\theta + \kappa\mathbf{D} + \kappa \sum_{r=2}^{\infty} (\mathbf{a}_r \cos r\theta + \mathbf{b}_r \sin r\theta) \\ \mathbf{k} &= \mathbf{C} + \kappa \sum_{r=1}^{\infty} (\mathbf{c}_r \cos r\theta + \mathbf{d}_r \sin r\theta) \end{aligned} \tag{20}$$

with

$$\begin{aligned} \frac{d\mathbf{A}}{d\theta} &= \kappa\mathbf{f}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \\ \frac{d\mathbf{B}}{d\theta} &= \kappa\mathbf{g}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \\ \frac{d\mathbf{C}}{d\theta} &= \kappa\mathbf{h}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \\ \frac{d\mathbf{D}}{d\theta} &= \kappa\mathbf{l}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) \end{aligned} \tag{21}$$

Moreover, the vectors $\mathbf{a}_r, \mathbf{b}_r, \mathbf{c}_r, \mathbf{d}_r$, $r = 2, 3, 4, \cdots$, are assumed to be slowly varying, so that $d\mathbf{a}_r/d\theta = \kappa\mathbf{f}_r$, etc. Neglecting κ^2 terms, it follows from Eq. (20) and (21) that

$$\begin{aligned} \frac{d\mathbf{i}}{d\theta} &= -\mathbf{A}\sin\theta + \mathbf{B}\cos\theta + \kappa\mathbf{f}\cos\theta \\ &\quad + \kappa\mathbf{g}\sin\theta - \kappa\sum_{2}^{\infty} r(\mathbf{a}_r \sin r\theta - \mathbf{b}_r \cos r\theta) \\ \frac{d^2\mathbf{i}}{d\theta^2} &= -\mathbf{A}\cos\theta - \mathbf{B}\sin\theta - 2\kappa\mathbf{f}\sin\theta + 2\kappa\mathbf{g}\cos\theta \\ &\quad - \kappa\sum_{2}^{\infty} r^2(\mathbf{a}_r \cos r\theta + \mathbf{b}_r \sin r\theta) \\ \frac{d\mathbf{k}}{d\theta} &= \kappa\mathbf{h} - \kappa\sum_{1}^{\infty} r(\mathbf{c}_r \sin r\theta - \mathbf{d}_r \cos r\theta) \end{aligned} \tag{22}$$

Substituting in Eq. (19) yields

$$\kappa \mathbf{D} - 2\kappa \mathbf{f} \sin\theta + 2\kappa \mathbf{g} \cos\theta + \kappa \sum_{2}^{\infty} (1 - r^2)(\mathbf{a}_r \cos r\theta - \mathbf{b}_r \sin r\theta)$$

$$= \frac{\kappa \mathbf{C}}{(1 + e\cos\theta)^3} = \kappa \mathbf{C}\left(\frac{\alpha_0}{2} + \sum_{r=1}^{\infty} \alpha_r \cos r\theta\right)$$

$$\kappa \mathbf{h} - \kappa \sum_{1}^{\infty} r(\mathbf{c}_r \sin r\theta - \mathbf{d}_r \cos r\theta) \qquad (23)$$

$$= \frac{\kappa}{(1 + e\cos\theta)^3}(\mathbf{A}\sin\theta - \mathbf{B}\cos\theta)$$

with

$$\alpha_r = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos r\theta}{(1 + e\cos\theta)^3}\, d\theta \qquad r = 0, 1, 2, \cdots$$

$$\mathbf{h} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathbf{A}\sin\theta - \mathbf{B}\cos\theta}{(1 + e\cos\theta)^3}\, d\theta \qquad (24)$$

$$= -\frac{\mathbf{B}}{2\pi}\int_0^{2\pi} \frac{\cos\theta}{(1 + e\cos\theta)^3}\, d\theta$$

The α_r, $r = 0, 1, 2, \cdots$, are the Fourier coefficients of $(1 + e\cos\theta)^{-3}$, and $\mathbf{h}$ is the steady-state component of $(\mathbf{A}\sin\theta - \mathbf{B}\cos\theta)/(1 + e\cos\theta)^3$.

From Eq. (23) it follows that

$$\mathbf{D} = \frac{\alpha_0}{2}\mathbf{C} = \frac{2 + e^2}{2}(1 - e^2)^{-5/2}\mathbf{C}$$

$$\mathbf{f} = 0$$

$$\mathbf{g} = \frac{\alpha_1}{2}\mathbf{C} = -\frac{3}{2}e(1 - e^2)^{-5/2}\mathbf{C}$$

$$\mathbf{b}_r = 0, r \geqq 2 \qquad (25)$$

$$\mathbf{a}_r = \frac{\mathbf{C}}{1 - r^2}\alpha_r, r \geqq 2$$

$$\mathbf{h} = +\frac{3}{2}e(1 - e^2)^{-5/2}\mathbf{B} = -\frac{\alpha_1}{2}\mathbf{B}$$

so that

$$\frac{d\mathbf{A}}{d\theta} = 0$$

$$\frac{d\mathbf{B}}{d\theta} = -\frac{3}{2}e(1 - e^2)^{-5/2}\kappa\mathbf{C} \qquad (26)$$

$$\frac{d\mathbf{C}}{d\theta} = +\frac{3}{2}e(1 - e^2)^{-5/2}\kappa\mathbf{B}$$

whose solutions are

$$\mathbf{A} = \mathbf{A}_0$$

$$\mathbf{B} = \mathbf{B}_0 \cos\frac{\kappa\alpha_1}{2}\theta + \mathbf{C}_0 \sin\frac{\kappa\alpha_1}{2}\theta \qquad (27)$$

$$\mathbf{C} = -\mathbf{B}_0 \sin\frac{\kappa\alpha_1}{2}\theta + \mathbf{C}_0 \cos\frac{\kappa\alpha_1}{2}\theta$$

with $\mathbf{A}_0$, $\mathbf{B}_0$, and $\mathbf{C}_0$ constant vectors of integration.

The values of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{a}_r$, and $\mathbf{b}_r$, as given by Eq. (25) and (27), determine the vector $\mathbf{i}$, and the motion is given by $\mathbf{r} = r\mathbf{i}$, with

$$r = \frac{\frac{h^2}{\mu}}{1 + e\cos\theta}$$

If e is set equal to zero, then the value for $\mathbf{i}$ reduces to the exact solution for the case of circular motion. This can be noted by comparing $\mathbf{i}$ with that $\mathbf{i}$ in Eq. (11), after replacing t by θ, α by 1, and β by κ.

V. ORBITS OF SMALL ECCENTRICITY

For $e << 1$ we can replace u^{-3} by the expression

$$u^{-3} \approx \frac{h^6}{\mu^3}(1 - 3e\cos\theta) \tag{28}$$

and Eq. (8) becomes

$$\begin{aligned} \frac{d\mathbf{i}}{d\theta} &= \mathbf{j} \\ \frac{d\mathbf{j}}{d\theta} &= -\mathbf{i} + \kappa(1 - 3e\cos\theta)\mathbf{k} \\ \frac{d\mathbf{k}}{d\theta} &= -\kappa(1 - 3e\cos\theta)\mathbf{j} \end{aligned} \tag{29}$$

where, neglecting e^2 terms,

$$\kappa = \frac{Wh^4}{\mu^3} = \frac{W\mu^2a^2}{\mu^3}(1 - e^2)^2 \approx \frac{W}{\frac{\mu}{a^2}}$$

The solutions of Eq. (29) for $e = 0$ are

$$\begin{aligned} \mathbf{i}_1 &= \kappa\mathbf{A} + \mathbf{B}\sin\omega\theta + \mathbf{C}\cos\omega\theta \\ \mathbf{j}_1 &= \omega\mathbf{B}\cos\omega\theta - \omega\mathbf{C}\sin\omega\theta \\ \mathbf{k}_1 &= \mathbf{A} - \kappa\mathbf{B}\sin\omega\theta - \kappa\mathbf{C}\cos\omega\theta \end{aligned} \tag{30}$$

where $\omega^2 = 1 + \kappa^2$, and **A**, **B**, and **C** are arbitrary constant vectors.

Next, we assume a solution of Eq. (29) of the form

$$\begin{aligned} \mathbf{i} &= \mathbf{i}_1 + e\mathbf{i}_2 \\ \mathbf{j} &= \mathbf{j}_1 + e\mathbf{j}_2 \\ \mathbf{k} &= \mathbf{k}_1 + e\mathbf{k}_2 \end{aligned} \tag{31}$$

Substituting the terms of Eq. (30) and (31) into Eq. (29), and neglecting e^2 terms, we obtain

$$\begin{aligned} \frac{d\mathbf{i}_2}{d\theta} &= \mathbf{j}_2 \\ \frac{d\mathbf{j}_2}{d\theta} &= -\mathbf{i}_2 + \kappa\mathbf{k}_2 - 3\kappa\cos\theta\,[\mathbf{A} - \kappa\mathbf{B}\sin\omega\theta - \kappa\mathbf{C}\cos\omega\theta] \\ \frac{d\mathbf{k}_2}{d\theta} &= -\kappa\mathbf{j}_2 + 3\omega\kappa\cos\theta\,[\mathbf{B}\cos\omega\theta - \mathbf{C}\sin\omega\theta] \end{aligned} \tag{32}$$

An integration of Eq. (32) yields

$$\begin{aligned} \mathbf{i} = {} & \kappa\mathbf{A} + \mathbf{B}\sin\omega\theta + \mathbf{C}\cos\omega\theta \\ & + \frac{3}{2}e\kappa^2\left\{\mathbf{B}\left[\frac{\sin(\omega-1)\theta}{\omega-1} - \frac{\sin(\omega+1)\theta}{\omega+1}\right]\right. \\ & + \mathbf{C}\left[\frac{\cos(\omega-1)\theta}{\omega-1} - \frac{\cos(\omega+1)\theta}{\omega+1}\right] \\ & \left. - \frac{2\mathbf{C}}{\kappa^2} + \frac{2\mathbf{A}}{\kappa^3}[\cos\omega\theta - \cos\theta]\right\} \end{aligned} \tag{33}$$

with similar expressions for **j** and **k**. For $\kappa = 0$, $\omega = 1$, the expression for **i** reduces to

$$\mathbf{i} = \mathbf{B}\sin\theta + \mathbf{C}\cos\theta \tag{34}$$

which is the solution of Eq. (29) with $\kappa = 0$.

With $\mathbf{i} = \mathbf{i}_0$, $\mathbf{j} = \mathbf{j}_0$, and $\mathbf{k} = \mathbf{k}_0$ for $\theta = 0$, we obtain

$$\begin{aligned} \mathbf{A} &= \frac{\kappa}{\omega^2}\mathbf{i}_0 + \frac{1}{\omega^2}\mathbf{k}_0 \\ \mathbf{B} &= \frac{1}{\omega}\mathbf{j}_0 \\ \mathbf{C} &= \frac{1}{\omega^2}\mathbf{i}_0 - \frac{\kappa}{\omega^2}\mathbf{k}_0 \end{aligned} \tag{35}$$

and the motion of the satellite is given by

$$\mathbf{r} = r\mathbf{i} \tag{36}$$

where **i** given by Eq. (33) and $r = h^2/\mu(1 - e\cos\theta) = a(1 - e\cos\theta)$, neglecting e^2 terms in r and in the product $r\mathbf{i}$. The motion is, of course, quite complicated since four fundamental frequencies occur: $\nu = 1$, ω, $\omega - 1$, and $\omega + 1$.

NOMENCLATURE

a	radius of circular orbit with $W = 0$
a_1	radius of circular orbit with $W \neq 0$
A, B, C	constants of integration, defined by Eq. (12)
d	distance of orbital plane from origin
e	eccentricity
h	z-component of angular momentum
H	height of satellite above X–Y plane
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors along positive x, y, and z axes
$\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0$	initial position vectors (in Sec. II at $\theta = 0$; subsequently, at $t = 0$)
O	origin of inverse-square-law force field
r	distance of satellite from origin
T	time to achieve the absolute maximum height from the X–Y plane
$u =$	$1/r$
$\mathbf{v}$	velocity of satellite
$W, \mathbf{W}$	thrust per unit mass on satellite
x, y, z	rotating coordinate frame of reference
X, Y, Z	inertial frame of reference
α, β	parameters of the motion, defined by Eq. (12)
θ	defined by $r^2\dot{\theta} = h$
κ	parameter of the motion
μ	proportionality constant of the inverse-square-law force field
ν	parameter of the motion, defined by Eq. (12)
ϕ	phase angle
$\omega, \boldsymbol{\omega}$	angular velocity

REFERENCES

1. Minorsky, N., *Introduction to Non-linear Mechanics,* J. W. Edwards Co., Ann Arbor, Michigan, 1947, pp. 183–243.

2. Lass, H., *Vector and Tensor Analysis,* McGraw Hill Co., New York, New York, 1950, pp. 58–61.